

Geometric quantization of weak-Hamiltonian functions

by

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ABSTRACT. The paper presents an extension of the geometric quantization procedure to integrable, big-isotropic structures. We obtain a generalization of the cohomology integrality condition, we discuss geometric structures on the total space of the corresponding principal circle bundle and we extend the notion of a polarization.

1 Big-isotropic structures

Weak-Hamiltonian functions belong to the framework of big-isotropic structures and have been discussed in [11, 12]. For the convenience of the reader, we recall some basic facts here.

All the manifolds and mappings are of C^∞ class and we denote by M an m -dimensional manifold, by $\chi^k(M)$ the space of k -vector fields, by $\Omega^k(M)$ the space of differential k -forms, by Γ the space of global cross sections of a vector bundle, by X, Y, \dots either contravariant vectors or vector fields, by α, β, \dots either covariant vectors or 1-forms, by d the exterior differential and by L the Lie derivative.

The vector bundle $T^{big}M = TM \oplus T^*M$ is called the *big tangent bundle*. It has the natural, non degenerate metric of zero signature (neutral metric)

$$g((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) + \beta(X)), \quad (1.1)$$

the non degenerate, skew-symmetric 2-form

$$\omega((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) - \beta(X)) \quad (1.2)$$

and the Courant bracket of cross sections

$$[(X, \alpha), (Y, \beta)]_C = ([X, Y], L_X\beta - L_Y\alpha + \frac{1}{2}d(\alpha(Y) - \beta(X))); \quad (1.3)$$

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unless to avoid confusion, the index C will be omitted.

Definition 1.1. A g -isotropic subbundle $E \subseteq T^{big}M$ of rank k ($0 \leq k \leq m$) is called a *big-isotropic structure* on M . A big-isotropic structure E is *integrable* if ΓE is closed by Courant brackets.

From the properties of the Courant bracket it follows that the integrability of E is equivalent with the property:

$$[\Gamma E, \Gamma E']_C \subseteq \Gamma E',$$

where $E' \perp_g E$, ($E \subseteq E'$) [11, 12]. The big-isotropic structures are a generalization of the almost Dirac structures (Dirac structures, in the integrable case), which are obtained for $k = m$.

Example 1.1. [11] Let $S \subseteq TM$ be a k -dimensional, regular distribution on M and $\lambda \in \Omega^2(M)$. Put

$$E_{(\lambda, S)} = \text{graph}(\flat_\lambda|_S) = \{(X, \flat_\lambda X = i(X)\lambda) / X \in S\}. \quad (1.4)$$

Then $E_{(\lambda, S)}$ is a big isotropic structure with the g -orthogonal bundle

$$E'_{(\lambda, S)} = \{(Y, \flat_\lambda Y + \gamma) / Y \in TM, \gamma \in \text{ann } S\}. \quad (1.5)$$

If $d\lambda = 0$, $E_{(\lambda, S)}$ is integrable iff S is a foliation.

Example 1.2. [11] Let Σ be a rank k subbundle of T^*M and $P \in \chi^2(M)$. Then

$$E_{(P, \Sigma)} = \text{graph}(\sharp_P|_\Sigma) = \{(\sharp_P \sigma = i(\sigma)P, \sigma) / \sigma \in \Sigma\} \quad (1.6)$$

is a big-isotropic structure on M with the g -orthogonal bundle

$$E'_{(P, \Sigma)} = \{(\sharp_P \beta + Y, \beta) / \beta \in T^*M, Y \in \text{ann } \Sigma\}. \quad (1.7)$$

If P is a Poisson bivector field the structure (1.6) is integrable iff Σ is closed with respect to the bracket of 1-forms defined by

$$\{\alpha, \beta\}_P = L_{\sharp_P \alpha} \beta - L_{\sharp_P \beta} \alpha - d(P(\alpha, \beta)). \quad (1.8)$$

For geometric quantization it is important to point out the existence of an adequate cohomology associated with an integrable big-isotropic structure E [11]. In the formulas below and in the remaining part of the paper, calligraphic letters denote pairs, $\mathcal{X} = (X, \alpha)$, $\mathcal{Y} = (Y, \beta)$, etc. The cochain spaces are the spaces of *truncated forms*

$$\Omega_{tr}^s(E) = \{\lambda : \wedge^{s-1} E \otimes E' \rightarrow C^\infty(M) / \lambda|_{\wedge^{s-1} E \otimes E} \in \wedge^s E^*\}. \quad (1.9)$$

The coboundary operator is defined by

$$d_{tr} \lambda(\mathcal{X}_1, \dots, \mathcal{X}_s, \mathcal{Y}) = \sum_{a=1}^s (-1)^{a+1} X_a(\lambda(\mathcal{X}_1, \dots, \hat{\mathcal{X}}_a, \dots, \mathcal{X}_s, \mathcal{Y}))$$

$$\begin{aligned}
& +(-1)^s Y(\lambda(\mathcal{X}_1, \dots, \mathcal{X}_s)) + \sum_{a < b=1}^s (-1)^{a+b} \lambda([\mathcal{X}_a, \mathcal{X}_b]_C, \mathcal{X}_1, \dots, \hat{\mathcal{X}}_a, \dots, \hat{\mathcal{X}}_b, \dots, \mathcal{X}_s, \mathcal{Y}) \\
& - \sum_{a=1}^s (-1)^a \lambda(\mathcal{X}_1, \dots, \hat{\mathcal{X}}_a, \dots, \mathcal{X}_s, [\mathcal{X}_s, \mathcal{Y}]_C),
\end{aligned}$$

where $\mathcal{X} \in \Gamma E, \mathcal{Y} \in \Gamma E'$. Since all the arguments that we use are pair-wise g -orthogonal, the Courant brackets above have the properties of the bracket of a Lie algebroid (the Jacobi identity in particular), therefore, the coboundary condition $d_{tr}^2 = 0$ holds. The corresponding cohomology spaces, called *truncated cohomology spaces*, will be denoted by $H_{tr}^s(E)$. The mapping $j : \Omega^s(M) \rightarrow \Omega_{tr}^s(E)$ defined by

$$(j\lambda)(\mathcal{X}_1, \dots, \mathcal{X}_{s-1}, \mathcal{Y}) = \lambda(X_1, \dots, X_{s-1}, Y)$$

is a morphism of cochain complexes, hence, there exist induced homomorphisms

$$j^* : H_{deR}^s(M, \mathbb{R}) \rightarrow H_{tr}^s(E).$$

Remark 1.1. In an appendix to this paper we will show that truncated cohomology has a further generalization in the case of a pair of Lie algebroids.

For the integrable big-isotropic structure E the form (1.2) defines the truncated 2-form $\omega_E = \omega|_{E \times E'}$ and a straightforward calculation gives $d_{tr}\omega_E = 0$, hence, one has a *fundamental cohomology class* $[\omega_E] \in H_{tr}^2(E)$ [11].

We end this section by explaining what is meant by a weak-Hamiltonian function with respect to an integrable, big-isotropic structure E (see details in [11]).

Definition 1.2. A function $f \in C^\infty(M)$ is a *Hamiltonian*, respectively *weak-Hamiltonian*, function if there exists a vector field $X_f \in \chi^1(M)$ such that $(X_f, df) \in \Gamma E$, respectively $(X_f, df) \in \Gamma E'$. The vector field X_f is a *Hamiltonian*, respectively *weak-Hamiltonian*, vector field of f .

The vector field X_f is required to be differentiable but it may not be unique. In the Hamiltonian case X_f is defined up to the addition of any $Z \in \chi^1(M) \cap \Gamma E$ and in the weak-Hamiltonian case up to $Z \in \chi^1(M) \cap \Gamma E'$. We denote by $C_{Ham}^\infty(M, E)$ the set of Hamiltonian functions, by $C_{wHam}^\infty(M, E)$ the set of weak-Hamiltonian functions and by $\mathcal{X}_f = (X_f, df)$ the pairs described in Definition 1.2.

Furthermore, if $f \in C_{Ham}^\infty(M, E)$ and $h \in C_{wHam}^\infty(M, E)$ one defines the *Poisson bracket*

$$\{f, h\} = -\omega_E(\mathcal{X}_f, \mathcal{X}_h) = X_f h = -X_h f, \quad (1.10)$$

which is easily seen not to depend on the choice of the Hamiltonian vector fields. The integrability of E implies $\{f, h\} \in C_{Ham}^\infty(M, E)$ and shows that one of the weak-Hamiltonian vector fields of the function $\{f, h\}$ is the Lie bracket $[X_f, X_h]$.

If both $f, h \in C_{Ham}^\infty(M, E)$, the Poisson bracket is skew symmetric and belongs to $C_{Ham}^\infty(M, E)$. The Poisson bracket satisfies the *Leibniz rule*

$$\{l, \{f, h\}\} = \{\{l, f\}, h\} + \{f, \{l, h\}\}, \quad (1.11)$$

$\forall l, f \in C_{Ham}^\infty(M, E), h \in C_{wHam}^\infty(M, E)$. Equality (1.11) restricts to the Jacobi identity on $C_{Ham}^\infty(M, E)$.

In the case of a Dirac structure discussed in [1], since $E' = E$, the notions of weak-Hamiltonian and Hamiltonian functions coincide.

Remark 1.2. The notion of an integrable big-isotropic structure may be complexified and all the previous result hold if complex values are assumed overall. Important examples are offered by the generalized CRF-structures, where E appears as the i -eigenbundle of a skew-symmetric endomorphism \mathcal{F} of $T^{big}M$ such that $\mathcal{F}^3 + \mathcal{F} = 0$ [13] and, in the Dirac case, by generalized complex structures, i.e., integrable i -eigenbundles of a skew-symmetric endomorphism \mathcal{I} of $T^{big}M$ such that $\mathcal{I}^2 = -Id$ [2].

2 Prequantization of big-isotropic structures

Prequantization is the first step of the geometric quantization procedure. It was defined and studied, independently, by J. M. Souriau [6] and B. Kostant [5] for symplectic manifolds, extended by several authors (e.g., [9]) to Poisson manifolds and by A. Weinstein and M. Zambon to Dirac manifolds [15]. Here, we extend prequantization further, to integrable big-isotropic structures. The path to follow is directly suggested by the Poisson and Dirac case.

Let E be an integrable big-isotropic structure on M and consider a triple (K, ∇, θ) where K is a Hermitian line bundle on M , ∇ is a Hermitian connection on K and θ is a truncated 1-cochain of E . Notice that the isomorphism \sharp_g (defined like in Riemannian geometry) yields an isomorphism $\Omega_{tr}^1(E) = E'^* \approx T^{big}M/E$, hence, the cochain θ may be seen as a pair $(U, \nu) \in \Gamma T^{big}M$ defined up to the addition of any $(X, \alpha) \in \Gamma E$. Our convention for this identification will be

$$\theta(Y, \beta) = \nu(Y) + \beta(U) \quad ((Y, \beta) \in E'). \quad (2.1)$$

The triple (K, ∇, θ) will be called a *g.p. (geometric prequantization) data system* if the *modified Kostant-Souriau formula*

$$\hat{h}s = \nabla_{X_h} s + 2\pi i(\theta(\mathcal{X}_h) + h)s \quad (s \in \Gamma K, h \in C_{wHam}^\infty(M, E)) \quad (2.2)$$

associates with every weak-Hamiltonian function h and every weak-Hamiltonian vector field X_h an operator $\hat{h} : \Gamma K \rightarrow \Gamma K$ with the following property: $\forall f \in C_{Ham}^\infty(M, E), h \in C_{wHam}^\infty(M, E)$ the commutant $[\hat{f}, \hat{h}] = \hat{f} \circ \hat{h} - \hat{h} \circ \hat{f}$ is equal to the operator $\widehat{\{f, h\}}$ associated to the Poisson bracket and to the choice $X_{\{f, h\}} = [X_f, X_h]$ of the weak-Hamiltonian vector field of $\{f, h\}$. In physics, the terminology is: *observable* for the function h and *quantum operator* for the operator \hat{h} and for other operators with a similar role.

Proposition 2.1. *A triple (K, ∇, θ) is a g.p. data system for E iff the curvature of the connection ∇ satisfies the following condition*

$$R_{\nabla}(X, Y) = 2\pi i(\omega_E(\mathcal{X}, \mathcal{Y}) - (d_{tr}\theta)(\mathcal{X}, \mathcal{Y})), \quad (2.3)$$

$$\forall \mathcal{X} = (X, \alpha) \in E_x, \mathcal{Y} = (Y, \beta) \in E'_x, x \in M.$$

Proof. A straightforward calculation gives

$$[\hat{f}, \hat{h}](s) = \widehat{\{f, h\}}s + R_{\nabla}(X_f, X_h)s + 2\pi i[(d_{tr}\theta)(\mathcal{X}_f, \mathcal{X}_h) + \{f, h\}]s,$$

where $\widehat{\{f, h\}}$ is constructed with the weak-Hamiltonian vector field $X_{\{f, h\}} = [X_f, X_h]$. Accordingly, we have a g.p. data system iff

$$R_{\nabla}(X_f, X_h) = -2\pi i(\{f, h\} + d_{tr}\theta(\mathcal{X}_f, \mathcal{X}_h)), \quad (2.4)$$

$\forall f \in C_{Ham}^{\infty}(M, E), h \in C_{wHam}^{\infty}(M, E)$, which is equivalent with the point-wise condition (2.3). The domains of the arguments in (2.3) are explained by the fact that $\forall (X, \alpha) \in E_x, (Y, \beta) \in E'_x$ there are functions $f \in C_{Ham}^{\infty}(M, E), h \in C_{wHam}^{\infty}(M, E)$ such that $d_x f = \alpha, d_x h = \beta, X_f(x) = X, X_h(x) = Y$. \square

The following result is an easy consequence of (2.3).

Proposition 2.2. *The integrable big-isotropic structure E admits g.p. data systems iff there exists a closed 2-form Θ on M which represents an integral de Rham cohomology class $[\Theta] \in H_{deR}^2(M)$ such that $j^*[\Theta] = [\omega_E]$.*

Proof. Since the 2-form $-(1/2\pi i)R_{\nabla}$ represents the first Chern class of K , (2.3) implies the required conclusion. Conversely, it is well known that if $-\Theta \in \Omega^2(M)$ represents an integral cohomology class $c \in H^2(M, \mathbb{Z})$ there exists a Hermitian line bundle K with the first Chern class c endowed with a Hermitian connection ∇ of curvature $2\pi i\Theta$. Furthermore, if $j^*[\Theta] = [\omega_E]$ there exists a 1-cochain $\theta \in \Omega_{tr}^1(M, E)$ such that

$$\Theta(X, Y) = \omega_E(\mathcal{X}, \mathcal{Y}) - (d_{tr}\theta)(\mathcal{X}, \mathcal{Y}), \quad (2.5)$$

which precisely is (2.3). \square

Remark 2.1. We may refer to the classification of the set of g.p. data systems like in [15]. Fix a cohomology class $c \in H^2(M, \mathbb{Z})$ with image $-\omega_E \in H_{tr}^2(M, E)$; then, the g.p. data systems (K, ∇, θ) where the line bundle K has first Chern class c are said to have topological type c . If the topological type is fixed, K is determined up to an isomorphism. Moreover, we may also consider that the corresponding 2-form Θ ($[\Theta] = -c$), therefore the connection ∇ too is fixed up to the previous isomorphism. Indeed, a change $\Theta \mapsto \Theta + d\xi$ is equivalent with a change of the cochain $\theta \mapsto \theta + j(\xi)$. But, if Θ is fixed, all the corresponding possible g.p. data systems are produced by all the 1-cochains θ that satisfy (2.5). Since condition (2.5) implies that $d_{tr}\theta$ is well defined, θ itself is defined up to the addition of a 1-cocycle $\kappa \in \Omega_{tr}^1(E)$, $d_{tr}\kappa = 0$. Therefore, the set of isomorphism classes of g.p. data systems of the topological type c is in a bijective correspondence with the set of d_{tr} -closed truncated 1-forms.

The expression (2.5) of the integrality condition may be put into the following simpler form.

Proposition 2.3. *The integrable big-isotropic structure E admits g.p. data systems iff there exists a closed 2-form Ξ on M that represents an integral de Rham cohomology class $[\Xi] \in H_{deR}^2(M)$ and a vector field $U \in \chi^1(M)$ such that*

$$\beta(X) + (L_U\beta)(X) - \alpha([Y, U]) = \Xi(X, Y), \quad \forall (X, \alpha) \in \Gamma E, (Y, \beta) \in \Gamma E'. \quad (2.6)$$

Proof. If the differential $d_{tr}\theta$ of the right hand side of (2.5) is replaced by its expression, while keeping in mind that $\mathcal{X} \perp_g \mathcal{Y}$, then (2.5) becomes (2.6) with $\Xi = -(\Theta + d\nu)$. \square

Remark 2.2. If we denote $L_U\mathcal{Y} = (L_UY, L_U\beta)$ condition (2.6) may be written under the form

$$\omega_E(\mathcal{X}, \mathcal{Y}) + 2g(\mathcal{X}, L_U\mathcal{Y}) = \Xi(X, Y), \quad \forall \mathcal{X} \in \Gamma E, \mathcal{Y} \in \Gamma E'. \quad (2.7)$$

Proposition 2.3 shows that the 1-form ν is not essential for prequantization. In fact, we have

Proposition 2.4. *If (K, ∇, θ) , where θ is defined by (2.1), is a g.p. data system for the integrable big-isotropic structure E then (K, ∇', θ') , where $\theta'(Y, \beta) = \beta(U)$, $\nabla' = \nabla + 2\pi i\nu$, also is a g.p. data system of E , which yields the same quantum operators (2.2) under the form*

$$\hat{h}s = \nabla'_{X_h}s + 2\pi i(U(h) + h)s \quad (s \in \Gamma K, h \in C_{wHam}^\infty(M, E)). \quad (2.8)$$

Moreover, $\forall \tilde{\nu} \in \Omega^1(M)$, $(K, \tilde{\nabla} = \nabla + 2\pi i(\nu - \tilde{\nu}), \tilde{\theta}(\mathcal{Y}) = \beta(U) + \tilde{\nu}(Y))$ also is a g.p. data system with the same quantum operators.

Proof. The new triples satisfy condition (2.3). \square

A cochain of the type $\theta'(Y, \beta) = \beta(U)$ will be called a *vectorial cochain* and Proposition 2.4 shows that it suffices to consider g.p. systems with vectorial cochains only. However, we will continue to write the formulas for arbitrary complex cochains θ .

In the case of a Dirac structure the conditions stated in Propositions 2.1, 2.2 coincide with those given in [15]. Below, we discuss the prequantization condition in Examples (1.1) and (1.2).

Example 2.1. Let $E_{(\lambda, S)}$ be the integrable big-isotropic structure associated with the closed 2-form λ and the foliation S of M . Then, the prequantization condition is (2.6) where

$$X \in S, \alpha = \flat_\lambda X, Y \in TM, \beta = \flat_\lambda Y + \gamma, \gamma \in \text{ann } S.$$

The corresponding result is

$$(\lambda + L_U\lambda)(X, Y) + \gamma([U, X]) = -\Xi(X, Y).$$

The case $Y = 0$ shows that, $\forall X \in S$, $[X, U] \in S$, i.e., U must be projectable onto the space of leaves of the foliation S . Furthermore, since $L_U \lambda = i(U)d\lambda + di(U)\lambda = di(U)\lambda$, the prequantization condition reduces to the fact that λ is a closed, integral 2-form. In particular, we see that $E_{(\lambda, S)}$ is prequantizable for any S using the cochain $\theta = 0$. The classical case of a symplectic manifold is included here.

Example 2.2. Let $E_{(P, \Sigma)}$ be the integrable big-isotropic structure associated with the Poisson bivector field P and the $\{, \}$ -closed subbundle $\Sigma \subseteq T^*M$. The prequantization condition is (2.6) where

$$\alpha \in \Sigma, X = \sharp_P \alpha, \beta \in T^*M, Y = \sharp_P \beta + Z, Z \in \text{ann } \Sigma$$

and a straightforward calculation gives

$$P(\alpha, \beta) - (L_U P)(\alpha, \beta) + \alpha([U, Z]) = \Xi(\sharp_P \alpha, \sharp_P \beta + Z), \quad (2.9)$$

where $U \in \chi^1(M)$ and Ξ is a closed, integral 2-form on M . For a Poisson structure $\Sigma = T^*M$ and $Z = 0$. Accordingly, (2.9) reduces to the known prequantization condition of a Poisson structure [9]. Furthermore, if P is defined by an integral symplectic form, the integrality condition is satisfied for any Σ if we take Ξ equal to the symplectic form and $U = 0$.

Remark 2.3. The quantum operators of physics act on a Hilbert space. We indicate the following procedure to transfer prequantization to a pre-Hilbert space [8, 9]; then, a corresponding Hilbert space can be constructed by completion. A complex half-density is a geometric object ρ with one complex component ρ_α with respect to local coordinates (x_α^i) on the coordinate neighborhood U_α such that on $U_\alpha \cap U_\beta$ one has

$$\rho_\beta = |\det(\partial x_\alpha^i / \partial x_\beta^j)|^{1/2} \rho_\alpha.$$

The complex half-densities define a line bundle $S^{1/2}$ on M . The Lie derivative acts on half-densities by

$$(L_X \rho)_\alpha = \xi_\alpha^i \frac{\partial \xi_\alpha^i}{\partial x_\alpha^i} + \frac{1}{2} \rho_\alpha \frac{\partial \rho_\alpha}{\partial x_\alpha^i} \quad (X = \xi^i \frac{\partial}{\partial x^i}),$$

where the Einstein summation convention is used. The quantum operators defined by (2.2) extend to $\Gamma(K \otimes S^{1/2}(M))$ by putting

$$\hat{h}(s \otimes \rho) = (\hat{h}s) \otimes \rho + s \otimes (L_{X_h} \rho). \quad (2.10)$$

Using bases of K and $S^{1/2}$, we see that any cross section $\sigma \in \Gamma(K \otimes S^{1/2}(M))$ has representations $\sigma = s \otimes \rho$ and that $\hat{h}\sigma$ is independent of the choice of the representation. Then, the space ${}^c\Gamma(K \otimes S^{1/2}(M))$, where the index c means that we take cross sections with a compact support, has the natural scalar product

$$\langle s_1 \otimes \rho_1, s_2 \otimes \rho_2 \rangle = \int_M \langle s_1, s_2 \rangle \rho_1 \bar{\rho}_2, \quad (2.11)$$

where $\langle s_1, s_2 \rangle$ is the Hermitian scalar product on K . The density version of Stoke's theorem (e.g., [8]) leads to the fact that the operators \hat{h} are antiunitary with respect to the product (2.11).

Remark 2.4. The results on prequantization also hold for complex big-isotropic structures, with the difference that the cohomology classes $[\omega_E], [\Theta], [\Xi]$ of the integrality condition are complex cohomology classes.

3 The prequantization space

Let E be an integrable big-isotropic structure on M and (K, ∇, θ) a g.p. data system. Let $p : Q \rightarrow M$ be a principal circle bundle such that K is associated to Q . Following [15], the total space Q will be called the *prequantization space*. In the case of a symplectic manifold, the prequantization space is a contact manifold and it was the basic object in Souriau's version of geometric quantization [6]. If $\|\cdot\|$ is the Hermitian norm on K , we may take

$$Q = \{b \in K / \|b\| = 1\}. \quad (3.1)$$

It is known that one has the following important geometric elements: 1) the vertical vector field $V \in \chi^1(Q)$ defined by the infinitesimal action of a basis of the Lie algebra $u(1)$ of S^1 by right translations, 2) the 1-form $\sigma \in \Omega^1(Q)$ of the principal bundle connection on Q that is equivalent with the covariant derivative ∇ ; the form σ vanishes on vectors that are horizontal with respect to the connection and $\sigma(V) = 1$.

We recall the definition of these elements. Take an open covering $M = \cup U_\alpha$ where K has the local unitary bases b_α and the transition functions

$$b_\beta = \gamma_{\alpha\beta} b_\alpha, \quad \gamma_{\alpha\beta} = e^{2\pi i \vartheta_{\alpha\beta}},$$

where we use $S^1 = \{e^{2\pi i t} / t \in \mathbb{R}\}$.

Then ∇ has the local equations $\nabla b_\alpha = \omega_\alpha b_\alpha$ where ω_α are the local connection forms and $\omega_\beta = \omega_\alpha + d\gamma_{\alpha\beta}$. The preservation of the Hermitian norm by ∇ implies that ω_α are purely imaginary forms and we denote $\omega_\alpha = 2\pi i \varpi_\alpha$.

The above description corresponds to the Lie algebra identification $u(1) = \text{span}_{\mathbb{R}}\{2\pi i\}$. If, instead, we take $u(1) = \mathbb{R}$, connection theory (e.g., see [4]) tells us that the form σ is defined by the formula

$$\sigma|_{p^{-1}(U_\alpha)} = p^* \varpi_\alpha + dt. \quad (3.2)$$

Then, the curvature form of the principal connection produced by ∇ on Q is $\Omega = p^*(d\varpi_\alpha)$ and one has

$$R_\nabla(X, Y) = -2\pi i \sigma([X^H, Y^H]), \quad (3.3)$$

where the upper index H denotes the horizontal lift with respect to the principal bundle connection. The horizontal lift $X^H(q)$, $q \in Q$, is defined for all $X \in T_x M$

($x \in M, p(q) = x$) and it is characterized by $p_*(X^H) = X, \sigma(X^H) = 0$. Finally, the expression (3.2) allows us to check that $V = \partial/\partial t$.

The importance of the prequantization space in geometric quantization comes from the following result (see [5] for the symplectic case):

Proposition 3.1. *Let $C_{inv}^\infty(Q, \mathbb{C})$ be the space of right-translation invariant, complex functions on Q . There exists a natural isomorphism of complex linear spaces $\Gamma K \approx C_{inv}^\infty(Q, \mathbb{C})$ that transposes the action of the quantum operator \hat{h} ($h \in C_{wHam}^\infty$) to the derivative defined by the vector field*

$$\bar{X}_h = X_h^H - [(\theta(\mathcal{X}_h) + h) \circ p]V \in \chi^1(Q). \quad (3.4)$$

Proof. The formula $s(p(q)) = \bar{s}(q)q$, where $q \in Q$ is seen as a basis of the fiber $K_{p(q)}$ and $s \in \Gamma K, \bar{s} \in C_{inv}^\infty(Q, \mathbb{C})$, defines an isomorphism $\Gamma K \approx C_{inv}^\infty(Q, \mathbb{C})$. This isomorphism sends the function $X_h^H \bar{s}$ to the cross section $\nabla_{X_h} s$ (Proposition III.1.3 of [4]) and a calculation via coordinates (x^i, t) , where (x^i) are coordinates on M , shows that $V\bar{s}$ corresponds to $-2\pi i s$. Hence, the operator (2.2) is transformed into (3.4). \square

Proposition 3.1 and property $\widehat{\{f, h\}} = [\hat{f}, \hat{h}]$ imply:

$$[\bar{X}_f, \bar{X}_h] = \bar{X}_{\{f, h\}}, \quad \forall f \in C_{Ham}^\infty(M, E), h \in C_{wHam}^\infty(M, E). \quad (3.5)$$

For instance, for the constant functions $f = 0, f = 1$ we may use $\mathcal{X}_f = (0, 0)$, which gives $\bar{X}_0 = 0, \bar{X}_1 = -V$ and

$$[X^H, V] \stackrel{(3.4)}{=} -[V, \bar{X}_h] \stackrel{(3.5)}{=} \bar{X}_{\{1, h\}} = \bar{X}_0 = 0. \quad (3.6)$$

Like in [15], we produce a geometric structure that accommodates the geometric elements defined by a g.p. data system on the prequantization space Q . This structure is defined on the stable tangent bundle [10]

$$\mathbf{T}^{big}Q = T^{big}Q \oplus \mathbb{R}^2 = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}).$$

For the vectors of $\mathbf{T}^{big}M$ we will use the following notation (pay attention to boldface characters)

$$\mathbf{X} = (\{X, u\}, \{\alpha, v\}) = (\mathcal{X}, u, v), \quad \mathcal{X} = (X, \alpha), u, v \in \mathbb{R}. \quad (3.7)$$

We recall that $\mathbf{T}^{big}M$ has the neutral metric

$$\mathbf{g}(\mathbf{X}_1, \mathbf{X}_2) = \frac{1}{2}(\alpha_1(X_2) + \alpha_2(X_1) + u_1v_2 + u_2v_1)$$

and the *Wade bracket* [14]

$$\begin{aligned} & [(\{X_1, u_1\}, \{\alpha_1, v_1\}), (\{X_2, u_2\}, \{\alpha_2, v_2\})]_W \\ &= (\{[X_1, X_2], X_1u_2 - X_2u_1\}, \{L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + \frac{1}{2}d(\alpha_1(X_2) - \alpha_2(X_1))\}) \end{aligned} \quad (3.8)$$

$$\begin{aligned}
& +u_1\alpha_2 - u_2\alpha_1 + \frac{1}{2}(v_2du_1 - v_1du_2 - u_1dv_2 + u_2dv_1), \\
& X_1v_2 - X_2v_1 + \frac{1}{2}(\alpha_1(X_2) - \alpha_2(X_1) - u_2v_1 + u_1v_2)\}.
\end{aligned}$$

Formula (3.4) suggests defining the *horizontal lift* of the bundle E by [15]:

$$E_q^H = \{(X^H - \theta(X, \alpha)V, p^*\alpha) / (X, \alpha) \in E_{p(q)}\} \subseteq T_q^{big}Q \ (q \in Q). \quad (3.9)$$

Obviously, E^H is a differentiable, big-isotropic bundle of the same rank as E . The connection with (3.4) is that every pair in E_q^H is the value of a pair $(\bar{X}_h, d(h \circ p))$ at q ; indeed, there exists a function h such that $(X_h, dh)_{p(q)} = (X, \alpha)$ and $h(p(q)) = 0$. Furthermore, we define the *stable lift* $\mathbf{E}^H \subseteq \mathbf{T}^{big}Q$ as follows

$$\mathbf{E}_q^H = \{(\{X^H - \theta(X, \alpha)V, 0\}, \{p^*\alpha, 0\}) / (X, \alpha) \in E_{p(q)}\} \oplus \text{span}\{\mathbf{V}\} \quad (3.10)$$

where $\mathbf{V} = (\{V, 0\}, \{0, 1\})$.

Proposition 3.2. \mathbf{E}^H is an integrable, isotropic subbundle of $\mathbf{T}^{big}Q$, $\text{rank}\mathbf{E}^H = \text{rank}E + 1$, with the \mathbf{g} -orthogonal subbundle

$$\mathbf{E}_q'^H = \{(\{Y^H - \theta(Y, \beta)V, 0\}, \{p^*\beta, 0\}) / (Y, \beta) \in E'_{p(q)}\} \quad (3.11)$$

$$\oplus \text{span}\{\mathbf{V}, (\{0, 0\}, \{0, 1\}), \mathbf{U} = (\{U^H, -1\}, \{\sigma + p^*\nu, 0\})\}.$$

Proof. The integrability of \mathbf{E}^H means the closure of $\Gamma\mathbf{E}^H$ with respect to the Wade bracket. We compute the Courant bracket of two cross sections of E^H :

$$\begin{aligned}
& [(X_1^H - \theta(X_1, \alpha_1)V, p^*\alpha_1), (X_2^H - \theta(X_2, \alpha_2)V, p^*\alpha_2)] \stackrel{(3.6)}{=} ([X_1^H, X_2^H] - X_1(\theta(X_2, \alpha_2))V \\
& + X_2(\theta(X_1, \alpha_1))V, L_{X_1^H - \theta(X_1, \alpha_1)V}(p^*\alpha_2) - L_{X_2^H - \theta(X_2, \alpha_2)V}(p^*\alpha_1) + p^*(d(\alpha_1(X_2))))).
\end{aligned}$$

If we express the Lie derivative by the Cartan formula $L_X = i(X)d + di(X)$, the T^*M -component becomes $p^*(L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + d(\alpha_1(X_2)))$. Then, since $[X_1, X_2]^H = pr_H[X_1^H, X_2^H]$, the TM -component is equal to

$$\begin{aligned}
& [X_1, X_2]^H + \sigma([X_1, X_2])V - (d_{tr}\theta)((X_1, \alpha_1), (X_2, \alpha_2))V - \theta([X_1, X_2])V \\
& \stackrel{(2.3), (3.3)}{=} [X_1, X_2]^H - \theta([X_1, \alpha_1], (X_2, \alpha_2))V + \omega_E((X_1, \alpha_1), (X_2, \alpha_2))V.
\end{aligned}$$

Now, for $\mathcal{X}_1, \mathcal{X}_2 \in \Gamma E^H$ we get

$$\begin{aligned}
& [(\mathcal{X}_1, 0, 0), (\mathcal{X}_2, 0, 0)]_W = [\mathcal{X}_1, \mathcal{X}_2]_C + (\{0, 0\}, \{0, \alpha_1(X_2)\}) \\
& = (([X_1, X_2]^H - \theta([X_1, \alpha_1], (X_2, \alpha_2))V, p^*(L_{X_1}\alpha_2 - L_{X_2}\alpha_1 \\
& + d(\alpha_1(X_2)))) , 0, 0) + \alpha_1(X_2)(\{V, 0\}, \{0, 1\}) \in \Gamma\mathbf{E}^H.
\end{aligned}$$

The proof of the integrability of \mathbf{E} is completed by the simple calculation

$$[\mathbf{V}, ((X^H + \lambda V, p^*\alpha), 0, 0)]_W = \mathbf{0}.$$

The proof of the other assertions of the proposition is straightforward. \square

Remark 3.1. Assume that the pair (U, ν) is g -isotropic (e.g., $\nu = 0$, see Proposition 2.4) and that E is a Dirac structure. Then, $\mathbf{E}^H \oplus \text{span}\{\mathbf{U}\}$ is a *Jacobi-Dirac structure* [15]. Indeed, it is easy to get $[\mathbf{U}, \mathbf{V}]_W = \mathbf{0}$. The only remaining condition $[((X^H - \theta(X, \alpha)V, p^*\alpha), 0, 0), \mathbf{U}]_W \in \Gamma \mathbf{E}^H$ can be deduced from properties of the Wade bracket. The latter is conformally related to a Courant bracket on $Q \times \mathbb{R}$ [10] and, if one proceeds like in Remark 1.1 of [10], one gets

$$\mathbf{g}([\mathbf{X}, \mathbf{X}_1]_W, \mathbf{X}_2) + \mathbf{g}(\mathbf{X}_1, [\mathbf{X}, \mathbf{X}_2]_W) = 0, \quad \forall \mathbf{X}, \mathbf{X}_1, \mathbf{X}_2 \in \Gamma \mathbf{E} \quad (3.12)$$

where \mathbf{E} is any almost Jacobi-Dirac structure on Q . Taking $\mathbf{X} = ((X^H - \theta(X, \alpha)V, p^*\alpha), 0, 0)$, $\mathbf{X}_1 = \mathbf{U}$ and, successively, $\mathbf{X}_2 = ((X'^H - \theta(X', \alpha')V, p^*\alpha'), 0, 0)$, $\mathbf{X}_2 = \mathbf{V}$, $\mathbf{X}_2 = \mathbf{U}$ in (3.12) we get $[((X^H - \theta(X, \alpha)V, p^*\alpha), 0, 0), \mathbf{U}]_W \perp_{\mathbf{g}} \mathbf{E}^H$. Since we are in the case where \mathbf{E}^H is maximal \mathbf{g} -isotropic we are done.

Another interesting, but less comprehensive, structure on Q is defined by the p -pullback of the structure E to Q , which turns out to be:

$$(p^*E)_q = \{(X^H + \lambda V, p^*\alpha) / (X, \alpha) \in E_{p(q)}\} = E^H \oplus \text{span}\{(V, 0)\}. \quad (3.13)$$

Obviously, p^*E is differentiable and $\text{rank } p^*E = \text{rank } E + 1$. Integrability of E and (3.6) imply that the bracket of two cross sections of E^H belongs to p^*E . Since we also have

$$[(V, 0), (X^H - \theta(X, \alpha)V, p^*\alpha)] = 0,$$

we see that p^*E is closed by Courant brackets.

Since we have

$$(L_V(X^H - \theta(X, \alpha)V), L_V(p^*\alpha)) = (0, 0), \quad (L_V V, L_V 0) = (0, 0),$$

the vector field V is an infinitesimal automorphism of p^*E . Accordingly, we may apply the prolongation construction of Theorem 2.1 of [10], which gives the subbundle

$$\tilde{E} = \text{span}\{(\{X^H + \lambda V, 0\}, \{p^*\alpha, 0\}), (\{0, 0\}, \{0, 1\})\} \subseteq \mathbf{T}^{big} M. \quad (3.14)$$

A straightforward calculation shows that \tilde{E} is integrable too.

Example 3.1. In the case of the structure $E_{(\lambda, S)}$ of Example 1.1 the pullback to Q is

$$p^*E_{(\lambda, S)} = \text{graph}(b_{p^*\lambda}|_{\text{span}\{Z^H, V\}, Z \in S}).$$

In the case of the structure $E_{(P, \Sigma)}$ of Example 1.2, if we consider the bivector field $\Pi = P^H + V \wedge U^H \in \chi^2(Q)$ and the morphism $\Psi : T^*Q \times \mathbb{R} \rightarrow TQ \times \mathbb{R}$ defined by

$$\Psi(\kappa, v) = (\sharp_\Pi \kappa + (v + P(\nu, \kappa))V, -\kappa(V)), \quad (\kappa \in T^*(Q), v \in \mathbb{R}),$$

one has

$$\mathbf{E}_{(P, \Sigma)}^H = \text{graph}(\Psi|_{p^*\Sigma \oplus \text{span}\{0, 1\}}).$$

4 Polarizations

In this section we discuss a problem that arises in the comparison of geometric prequantization with quantization commonly used in physics.

This discussion is motivated on one hand by the necessity to remove the ambiguity of the quantum operator due to the non-uniqueness of the (weak)-Hamiltonian vector field and on the other hand by the following example.

The dynamics of a mechanical system with holonomic constraints may be defined by a weak-Hamiltonian vector field with respect to an integrable, big-isotropic structure of the type $E_{(P,\Sigma)}$. Namely [12], assume that the configuration space of the system is the manifold N with local coordinates (q^i) , the phase space is $M = T^*N$ with canonical, local coordinates (q^i, p_i) and the constraints are defined by the regular, integrable distribution $L \subseteq TN$. In what follows we use the Einstein summation convention.

Take

$$P = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}, \quad \Sigma = \text{ann } \sharp_P(\pi^* \text{ann } L),$$

where $\pi : M \rightarrow N$ is the natural projection. Since P is defined by the exact symplectic form $\omega = dq^i \wedge dp_i \in \Omega^2(M)$, the prequantization condition (2.9) is satisfied. More exactly, we may use a g.p. data system given by the trivial bundle K with basis 1 and metric $\|1\| = 1$, the connection ∇ defined by $\nabla 1 = 2\pi i(p_i dq^i)1$ and the cochain $\theta = 0$. Then, formula (1.7) shows that the Hamiltonian vector fields of the function q^i are

$$X_{q^i} = \frac{\partial}{\partial p_i} + \alpha_a \varphi_i^a \frac{\partial}{\partial p_i}$$

where $\varphi_i^a dq^i = 0$ are the (independent) equations of L . The corresponding quantum operator is

$$\hat{q}^i(s1) = [\frac{\partial s}{\partial p_i} + \alpha_a \varphi_i^a \frac{\partial s}{\partial p_i} + 2\pi i q^i s]1.$$

The result is unambiguous and reduces to multiplication by $2\pi i q^i$ as required by physics¹ if $\partial s / \partial p_i = 0$.

In symplectic geometry, the distribution $\text{span}\{\partial/\partial p_i\}$ is called a polarization, and we want a corresponding notion in the general case. We will extend the definition that we gave in [9] for Poisson manifolds.

Definition 4.1. A *real polarization* of an integrable, big-isotropic structure E is a pair of subspaces $\mathcal{P} \subseteq \Gamma E, \mathcal{P}' \subseteq \Gamma E'$ with the following properties

$$\begin{aligned} \mathcal{P} &\subseteq \mathcal{P}', \quad \chi^1(M) \cap \Gamma E \subseteq \mathcal{P}, \quad \chi^1(M) \cap \Gamma E' \subseteq \mathcal{P}', \\ [\mathcal{P}, \mathcal{P}]_C &\subseteq \mathcal{P}, \quad [\mathcal{P}, \mathcal{P}']_C \subseteq \mathcal{P}', \quad \omega|_{\mathcal{P} \times \mathcal{P}'} = 0. \end{aligned} \tag{4.1}$$

¹In fact, physics requires multiplication by q^i , which happens if we divide the Kostant-Souriau formula by $2\pi i$.

A *complex polarization* of (E, E') is defined in the same way but replacing $E, E', \chi^1(M)$ by their complexifications $E_c = E \otimes \mathbb{C}, E'_c = E' \otimes \mathbb{C}, \chi_c^1(M) = \Gamma T_c M, T_c M = TM \otimes \mathbb{C}$.

For the simplicity of notation we refer to real polarizations but the results hold for complex polarizations as well.

With a polarization, we can associate the subspaces

$$\Gamma_{\mathcal{P}} K = \{s \in \Gamma K / \nabla_Y s = -2\pi i \theta(Y, \beta)s, \forall (Y, \beta) \in \mathcal{P}\},$$

$$\Gamma_{\mathcal{P}'} K = \{s \in \Gamma K / \nabla_Z s = -2\pi i \theta(Z, \zeta)s, \forall (Z, \zeta) \in \mathcal{P}'\},$$

which satisfy the condition $\Gamma_{\mathcal{P}'} K \subseteq \Gamma_{\mathcal{P}} K$, as well as the space of *polarized Hamiltonians*

$$C_{Ham}^\infty(M, \mathcal{P}, \mathcal{P}') = \{f \in C_{Ham}^\infty(M, E) / [(X_f, df), (Z, \zeta)] \in \mathcal{P}', \forall (Z, \zeta) \in \mathcal{P}'\}$$

and the space of *polarized weak-Hamiltonians*

$$C_{wHam}^\infty(M, \mathcal{P}, \mathcal{P}') = \{h \in C_{wHam}^\infty(M, E) / [(Y, \beta), (X_h, dh)] \in \mathcal{P}', \forall (Y, \beta) \in \mathcal{P}\}.$$

In the case of a complex polarization the previous spaces will be assumed to consist of complex valued functions.

Example 4.1. The pair $\mathcal{P} = \chi^1(M) \cap \Gamma E, \mathcal{P}' = \chi^1(M) \cap \Gamma E'$ is a polarization such that the corresponding spaces $\Gamma_{\mathcal{P}} K, \Gamma_{\mathcal{P}'} K$ are

$$\Gamma_E K = \{s \in \Gamma K / \nabla_Y s = -2\pi i \theta(Y, 0)s, \forall Y \in \chi^1(M) \cap \Gamma E\},$$

$$\Gamma_{E'} K = \{s \in \Gamma K / \nabla_Z s = -2\pi i \theta(Z, 0)s, \forall Z \in \chi^1(M) \cap \Gamma E'\},$$

respectively, and

$$C_{Ham}^\infty(M, \mathcal{P}, \mathcal{P}') = C_{Ham}^\infty(M, E), C_{wHam}^\infty(M, \mathcal{P}, \mathcal{P}') = C_{wHam}^\infty(M, E).$$

The restrictions $\hat{h}|_{\Gamma_{E'} K}, \hat{f}|_{\Gamma_E K}$ where $h \in C_{wHam}^\infty(M, E), f \in C_{Ham}^\infty(M, E)$ are independent of the chosen Hamiltonian vector fields. This polarization does not solve the difficulty indicated by the example of the constrained mechanical systems and there is a need for bigger, preferably maximal, polarizations.

Example 4.2. Let $E = \text{graph } \sharp_\Pi$ be the Dirac structure associated with a Poisson bivector field Π . Then there exists a bijective correspondence between the complex polarizations $(\mathcal{P}, \mathcal{P}' = \mathcal{P})$ and the subalgebras \mathcal{Q} of the Lie algebra $(\Omega^1 \otimes \mathbb{C}, \{., .\}_\Pi)$ (such a subalgebra defined the notion of a polarization in [9]). This correspondence is given by $\mathcal{P} \mapsto \mathcal{Q} = \text{pr}_{\Omega^1(M)} \mathcal{P}$. Furthermore, the space of polarized Hamiltonians is given by [9]

$$C_{Ham}^\infty(M, \mathcal{P}, \mathcal{P}) = \{f \in C^\infty(M, \mathbb{C}) / \{df, \alpha\}_\Pi \in \mathcal{Q}, \forall \alpha \in \mathcal{Q}\}.$$

Finally, if Π is prequantizable by a Hermitian line bundle K and a contravariant derivative D [9], we may take an arbitrary Hermitian connection ∇ on K and define a cochain by the formula

$$\theta(\sharp_\Pi \xi, \xi)s = \frac{1}{2\pi i}(D_\xi s - \nabla_{\sharp_\Pi \xi} s), \quad s \in \Gamma K.$$

Then, (K, ∇, θ) is a g.p. data system and one has [9]

$$\Gamma_{\mathcal{P}} K = \{s \in \Gamma K / D_\xi s = 0, \forall \xi \in \mathcal{Q}\}.$$

For instance, if $M = \mathbb{R}^{2n+h} = \{(q^i, p_i, t^u)\}$ and

$$\Pi = \sum_{i=1}^n \varphi_i(t^u) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$$

then $\mathcal{Q} = \text{span}\{\varphi_i(t^u) dq^i\}$ (no summation) produces a polarization \mathcal{P} .

Now, we can prove the following proposition.

Proposition 4.1. *For any functions $h \in C_{wHam}^\infty(M, \mathcal{P}, \mathcal{P}')$, $f \in C_{Ham}^\infty(M, \mathcal{P}, \mathcal{P}')$, the operators \hat{h} restrict to a unique operator $\hat{h} : \Gamma_{\mathcal{P}'} K \rightarrow \Gamma_{\mathcal{P}} K$ and the operators \hat{f} restrict to a unique operator $\hat{f} : \Gamma_{\mathcal{P}'} K \rightarrow \Gamma_{\mathcal{P}'} K$.*

Proof. The operators $\hat{h}|_{\Gamma_{\mathcal{P}'} K}$, $\hat{f}|_{\Gamma_{\mathcal{P}'} K}$ are independent of the choice of the Hamiltonian vector fields required by (2.2) because of the second and third conditions (4.1) and since $\Gamma_{\mathcal{P}'} K \subseteq \Gamma_{\mathcal{P}} K$. Consider a field $(Y, \beta) \in \mathcal{P}$ and a cross section $s \in \Gamma_{\mathcal{P}'} K$. Then,

$$\begin{aligned} \nabla_Y(\hat{h}s) &= \nabla_Y \nabla_{X_h} s + 2\pi i Y[(\theta(\mathcal{X}_h) + h)s + 2\pi i[\theta(\mathcal{X}_h) + h]\nabla_Y s \\ &= R_\nabla(Y, X_h)s + \hat{h}(\nabla_Y s) + \nabla_{[Y, X_h]} s + 2\pi i[Y(\theta(\mathcal{X}_h)) + Yh]s. \end{aligned}$$

In the result we insert

$$\nabla_Y s = -2\pi i \theta(Y, \beta)s, \quad \nabla_{[Y, X_h]} s = -2\pi i \theta([Y, \beta], (X_h, dh))s$$

(the second equality is implied by the definition of $C_{wHam}^\infty(M, \mathcal{P}, \mathcal{P}')$) and

$$R_\nabla(Y, X_h) \stackrel{(2.3)}{=} -2\pi i[d_{tr}\theta((Y, \beta), (X_h, dh)) - \omega_E((Y, \beta), (X_h, dh))],$$

where, in fact,

$$\omega_E((Y, \beta), (X_h, dh)) = -(Yh),$$

because the arguments are g -orthogonal $((Y, \beta) \in \mathcal{P} \subseteq \Gamma E, (X_h, dh) \in \Gamma E')$. Then, after reductions, we get the required result:

$$\nabla_Y(\hat{h}s) = -2\pi i \theta(Y, \beta)(\hat{h}s).$$

The same calculation for $f \in C_{Ham}^\infty(M, \mathcal{P}, \mathcal{P}')$, $s \in \Gamma_{\mathcal{P}'} K$, $(Y, \beta) \in \mathcal{P}'$ yields the second conclusion. \square

Remark 4.1. The condition $\omega|_{\mathcal{P} \times \mathcal{P}'} = 0$ that enters in (4.1) played no role in the previous proof. However, it must be imposed because it is a necessary condition for $\Gamma_{\mathcal{P}}K \neq 0, \Gamma_{\mathcal{P}'}K \neq 0$. This follows by using (2.3) for $(Y, \beta) \in \mathcal{P}, (Z, \zeta) \in \mathcal{P}', s \in \Gamma_{\mathcal{P}'}K \subseteq \Gamma_{\mathcal{P}}K$. However, this condition is not sufficient and $\Gamma_{\mathcal{P}} \neq 0, \Gamma_{\mathcal{P}'} \neq 0$ have to be assumed.

If the polarization is of the form $\mathcal{P} = \Gamma P, \mathcal{P}' = \Gamma P'$ where P, P' are subbundles of E, E' , respectively, we have

$$\Gamma_{\mathcal{P}}K = \Gamma_P K = \{s \in \Gamma K / \nabla_{Y_x} s = -2\pi i \theta(Y_x, \beta_x)s, \forall (Y_x, \beta_x) \in P_x, x \in M\},$$

$$\Gamma_{\mathcal{P}'}K = \Gamma_{P'} K = \{s \in \Gamma K / \nabla_{Z_x} s = -2\pi i \theta(Z_x, \zeta_x)s, \forall (Z_x, \zeta_x) \in P'_x, x \in M\},$$

since $(\nabla_Y s)(x) = \nabla_{Y_x} s$, etc. and any $(Y_x, \beta_x) \in P_x, (Z_x, \zeta_x) \in P'_x$ have global, differentiable extensions in $\Gamma P, \Gamma P'$. This point-wise setting can be extended as follows.

Recall that the distribution $\mathcal{E} = \text{pr}_{TM} E$ is a generalized foliation that defines the characteristic leaves \mathcal{L} of E [11]. Any cross sections $\mathcal{X} \in \Gamma(E|_{\mathcal{L}}), \mathcal{Y} \in \Gamma(E'|_{\mathcal{L}})$ have differentiable extensions say, $\tilde{\mathcal{X}} \in \Gamma E, \tilde{\mathcal{Y}} \in \Gamma E'$ to M (at least locally) and we can define a bracket

$$[\mathcal{X}, \mathcal{Y}] = [\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}]|_{\mathcal{L}} \in \Gamma(E'|_{\mathcal{L}}), \quad (4.2)$$

which is independent on the choice of the extensions. Indeed (like in the proof of Theorem 2.1 of [3]), it suffices to show that the bracket vanishes for $\tilde{\mathcal{Y}} = \sum_i \tilde{\lambda}^i \tilde{\mathcal{B}}_i$ where $\tilde{\mathcal{B}}_i$ is a local basis of E' and $\tilde{\lambda}^i|_{\mathcal{L}} = 0$ (we do not have to consider a similar $\tilde{\mathcal{X}}$ because $E \subseteq E'$.) Since for g -orthogonal arguments the Courant bracket behaves like a Lie algebroid bracket, we have

$$[\tilde{\mathcal{X}}, \sum_i \tilde{\lambda}^i \tilde{\mathcal{B}}_i] = \sum_i \{\tilde{\lambda}^i [\tilde{\mathcal{X}}, \tilde{\mathcal{B}}_i] + ((\text{pr}_{TM} \tilde{\mathcal{X}}) \tilde{\lambda}^i) \tilde{\mathcal{B}}_i\},$$

which has the zero restriction to \mathcal{L} because $\text{pr}_{TM} \mathcal{X} \in \mathcal{L}$.

Accordingly, if we denote by $\bar{\Gamma} E, \bar{\Gamma} E'$ the spaces of (possibly non differentiable) cross sections of E, E' that have differentiable restrictions to each leaf \mathcal{L} then, we get a bracket

$$[\bar{\mathcal{X}}, \bar{\mathcal{Y}}] \in \bar{\Gamma} E', \forall \bar{\mathcal{X}} \in \bar{\Gamma} E, \bar{\mathcal{Y}} \in \bar{\Gamma} E',$$

which belongs to $\bar{\Gamma} E$ if $\bar{\mathcal{X}}, \bar{\mathcal{Y}} \in \bar{\Gamma} E$.

By generalized subbundles $P \subseteq E, P' \subseteq E'$ we will understand fields of subspaces of the fibers of E, E' such that the restrictions to each leaf \mathcal{L} are differentiable vector bundles along \mathcal{L} . For instance, $TM \cap E, TM \cap E'$ are generalized subbundles of E, E' , respectively. We shall use the notation $\bar{\Gamma} P, \bar{\Gamma} P'$ in the same sense as for E, E' .

Definition 4.2. A *real point-wise polarization* of an integrable, big-isotropic structure E is a pair of generalized subbundles $P \subseteq E, P' \subseteq E'$ with the following properties

$$\begin{aligned} P &\subseteq P', TM \cap E \subseteq P, TM \cap E' \subseteq P', \\ [\bar{\Gamma} P, \bar{\Gamma} P] &\subseteq \bar{\Gamma} P, [\bar{\Gamma} P, \bar{\Gamma} P'] \subseteq \bar{\Gamma} P', \omega|_{P \times P'} = 0. \end{aligned} \quad (4.3)$$

A *complex point-wise polarization* is defined in the same way using the complexified bundles $E_c, E'_c, T_c M$.

If (P, P') is a point-wise polarization, the spaces $\Gamma_P K, \Gamma_{P'} K$ may still be defined and we may also define the following spaces of functions

$$C_{Ham}^\infty(M, P, P') = \{f \in C_{Ham}^\infty(M, E) / [(X_f, df), (Z, \zeta)] \in \bar{\Gamma} P', \forall (Z, \zeta) \in \bar{\Gamma} P'\},$$

$$C_{wHam}^\infty(M, P, P') = \{h \in C_{wHam}^\infty(M, E) / [(Y, \beta), (X_h, dh)] \in \bar{\Gamma} P', \forall (Y, \beta) \in \bar{\Gamma} P'\}.$$

With this notation we get

Proposition 4.2. *For any function $f \in C_{Ham}^\infty(M, P, P')$, the operators \hat{f} restrict to a well defined operator $\hat{f} : \Gamma_{P'} K \rightarrow \Gamma_{P'} K$.*

Proof. The uniqueness of $\hat{f}|_{\Gamma_{P'} K \supseteq \Gamma_{P'} K}$ follows from the second condition (4.3). Take $(Z_x, \zeta_x) \in P'_x$ ($x \in M$) and extend it to a differentiable cross section $(Z, \zeta) \in \Gamma(P'|_{\mathcal{L}_x})$, where \mathcal{L}_x is the characteristic leaf of E through x . Let $(\tilde{Z}, \tilde{\zeta}) \in \Gamma E$ be a further extension of (Z, ζ) to a neighborhood of x in M . Then, for $s \in \Gamma_{P'} K$ one has

$$\begin{aligned} \nabla_{Z_x}(\hat{f}s) &= \nabla_{Z_x} \nabla_{X_f} s + 2\pi i Z_x[(\theta(\mathcal{X}_f)) + f]s \\ &\quad + 2\pi i[\theta(\mathcal{X}_f) + f](x) \nabla_{Z_x} s. \end{aligned} \tag{4.4}$$

On the other hand, one has

$$R_\nabla(Z_x, X_f(x))s = \nabla_{Z_x} \nabla_{X_f} s - \nabla_{X_f(x)} \nabla_{\tilde{Z}} s - \nabla_{[\tilde{Z}, X_f](x)} s,$$

where the following happen: 1) since $X_f(x)$ is tangent to \mathcal{L}_x , $\nabla_{X_f(x)} \nabla_{\tilde{Z}} s$ depends only on $\nabla_{\tilde{Z}} s|_{\mathcal{L}_x}$, which is equal to $-2\pi i \theta(Z, \zeta)s|_{\mathcal{L}_x}$ by the definition of $\Gamma_{P'} K$; 2) since $[(\tilde{Z}, \tilde{\zeta}), (X_f, df)](x) = -[(X_f, df)|_{\mathcal{L}_x}, (Z, \zeta)](x)$, the definitions of $C_{Ham}^\infty(M, P, P')$ and $\Gamma_{P'} K$ imply

$$\nabla_{[\tilde{Z}, X_f](x)} s = 2\pi i \theta_x([(X_f, df)|_{\mathcal{L}_x}, (Z, \zeta)](x))s(x).$$

If these results are inserted in (4.4) the same reductions like in the proof of Proposition 4.1 hold and one gets $\hat{f}s \in \Gamma_{P'} K$. \square

Remark 4.2. In the case of a Dirac structure it is natural to consider only polarizations with $\mathcal{P}' = \mathcal{P}$, $P' = P$, respectively. Accordingly the definitions will not refer to \mathcal{P}', P' any more and Proposition 4.2 extends Lemma 6.1 of [15], which was proven differently there.

Example 4.3. Consider the integrable, big-isotropic structure $E_{(\lambda, S)}$ of Example 1.1, where λ is a closed 2-form and S is an involutive subbundle of TM . If $S = TM$, E is the Dirac structure defined by the presymplectic form λ , $TM \cap E = \ker \lambda$ and one has only one characteristic leaf $\mathcal{L} = M$. In this case, the examination of conditions (4.3) is easy and shows that a point-wise polarization with $P' = P$ may be identified with a vector subbundle $L = \text{pr}_{TM} P$ such

that $\ker \lambda \subseteq L$, $\lambda|_L = 0$ and L is involutive. In the symplectic case, $\ker \lambda = 0$ and, if we ask maximality of P , L is an involutive, Lagrangian subbundle as required by the classical definition of a polarization of a symplectic manifold. In both cases one gets

$$C_{Ham}^\infty(M, P, P') = \{f \in C_{Ham}^\infty(M, E) / [X_f, Y] \in \Gamma L\}. \quad (4.5)$$

Then, if λ is an integral form and we take the g.p. data system $(K, \nabla, 0)$ (see Example 2.1), we have

$$\Gamma_P K = \{s \in \Gamma K / \nabla_Y s = 0, \forall Y \in L\}. \quad (4.6)$$

For $S \subset TM$, the characteristic leaves are the leaves of the foliation S , which is regular, hence, it is natural to look for point-wise polarizations defined by regular subbundles P, P' . Necessarily, $P = \text{graph}(b_\lambda|_L)$ where $L = \text{pr}_{TM} P$ is an involutive subbundle of S such that $\lambda|_L = 0$ and $TM \cap E_{(\lambda, S)} = \ker \lambda \cap S \subseteq L$. Formula (1.5) implies $TM \cap E'_{(\lambda, S)} = S^{\perp_\lambda}$ and P' has to be a convenient enlargement of P such that $S^{\perp_\lambda} \subseteq P'$.

Assume that we are in the particular case where $\lambda|_S$ is non degenerate, hence, it induces symplectic forms of the leaves of S . Then, $S \cap S^{\perp_\lambda} = 0$ and, since $\ker \lambda \subseteq S^{\perp_\lambda}$, $TM \cap E_{(\lambda, S)} = 0$. If we start with a Lagrangian subfoliation L of $(S, \lambda|_S)$, we obtain a subbundle $P = \text{graph}(b_\lambda|_L)$ as required and the addition of

$$P' = \{(X + Y, b_\lambda X + b_\lambda Y + \gamma) / X \in L, Y \in S^{\perp_\lambda}, \gamma \in \text{ann } S\}$$

(we might have omitted $b_\lambda Y$ that belongs to $\text{ann } S$, but, it is convenient to keep it for the calculation that follows) gives a polarization. Indeed, we have $S^{\perp_\lambda} \subseteq P'$ (take $X = 0, \gamma = -b_\lambda Y$), $[\Gamma P, \Gamma P] \subseteq \Gamma P$ and $\omega|_{P \times P'} = 0$. For the closure condition $[\Gamma P, \Gamma P'] \subseteq \Gamma P'$, it suffices to check

$$\begin{aligned} [(X, b_\lambda X), (Y, b_\lambda Y)] &= ([X, Y], b_\lambda [X, Y]) \in \Gamma P', \\ [(X, b_\lambda X), (0, \gamma)] &= (0, L_X \gamma) \in \Gamma P', \end{aligned} \quad (4.7)$$

where Y is L -foliated (since the pairs in the left hand sides of (4.7) locally span $\Gamma P'$). Then $[X, Y] \in L$ and the first relation (4.7) holds with P' replaced by P . Finally, it is easy to check that $(0, L_X \gamma) \in \text{ann } S \subseteq P'$, which proves the second relation (4.7).

Formula (4.6) obviously remains valid. Formula (4.5) remains valid too. Indeed, using the fact that the infinitesimal transformation X_f preserves S we get

$$C_{Ham}^\infty(M, P, P') = \{f \in C_{Ham}^\infty(M, E) / [X_f, Z + Y] \in \Gamma(L \oplus S^{\perp_\lambda})\}$$

for $Z \in \Gamma L, Y \in \Gamma S^{\perp_\lambda}$. Since $L_{X_f} \lambda = di(X_f) \lambda = d^2 f = 0$, X_f preserves S^{\perp_λ} too and $[X_f, Y] \in \Gamma S^{\perp_\lambda}$. Therefore, we remain with the condition $[X_f, Z] \in \Gamma L$ that appears in (4.5). Furthermore, it follows straightforwardly that

$$C_{wHam}^\infty(M, P, P') = \{h \in C_{wHam}^\infty(M, E) / [X, X_h] \in \Gamma(L \oplus S^{\perp_\lambda}), \forall X \in \Gamma L\}.$$

The simplest example of the situation discussed above is given by

$$M = \mathbb{R}^{2n+h} = \{(q^i, p_i, t^u)\}, \quad \lambda = \sum_{i=1}^n dq^i \wedge dp_i,$$

$$S = \text{span}\left\{\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i}\right\}, \quad S^{\perp_\lambda} = \text{span}\left\{\frac{\partial}{\partial t^u}\right\}, \quad L = \text{span}\left\{\frac{\partial}{\partial p_i}\right\}.$$

Then,

$$P = \text{span}\left\{\left(\frac{\partial}{\partial p_i}, -dq^i\right)\right\}, \quad P' = P \oplus \text{span}\left\{\left(\frac{\partial}{\partial t^u}, 0\right)\right\} \oplus \text{span}\{(0, dt^u)\}$$

define a polarization.

Example 4.4. This example extends the classical notion of a Kähler polarization to generalized geometry. We briefly recall the framework following [2, 13]. A classical metric F-structure on a manifold M is a pair (F, γ) where γ is a Riemannian metric, $F \in \text{End} TM$, $F^3 + F = 0$ and

$$\gamma(FX, Y) + \gamma(X, FY) = 0 \Leftrightarrow b_\gamma \circ F = -{}^t F \circ b_\gamma \quad (4.8)$$

(t denotes transposition). A generalized metric F-structure is an analogous structure on $T^{big}M$ and is equivalent with a system that consists of two classical metric F-structures with the same metric, (γ, F_+, F_-) and a 2-form $\psi \in \Omega^2(M)$. Then, there are two injections $j_\pm : TM \rightarrow T^{big}M$ defined by

$$j_+(X) = (X, b_{\psi+\gamma}X), \quad j_-(X) = (X, b_{\psi-\gamma}X) \quad (X \in TM) \quad (4.9)$$

such that $T^{big}M = \text{im } j_+ \oplus \text{im } j_-$, where $\text{im } j_+ \perp_g \text{im } j_-$ and

$$g((X, b_{\psi\pm\gamma}X), (Y, b_{\psi\pm\gamma}Y)) = \pm\gamma(X, Y). \quad (4.10)$$

The injections j_\pm send F_\pm to structures \mathcal{F}_\pm on their images with $\mathcal{F} = \mathcal{F}_+ + \mathcal{F}_- \in \text{End}(T^{big}M)$ such that $\mathcal{F}^3 + \mathcal{F} = 0$.

The structures F_\pm yield decompositions

$$T_c M = H_\pm \oplus \bar{H}_\pm \oplus Q_{\pm c} \quad (Q_{\pm c} = Q_\pm \times \mathbb{C}, \quad Q_\pm \subseteq TM)$$

where the terms are the $(\pm i, 0)$ -eigenbundles, respectively. If we denote $E_\pm = j_\pm(H_\pm)$, $S_\pm = j_\pm(Q_\pm)$, (4.8) and (4.10) imply that

$$E_1 = E_+ \oplus E_-, \quad E_2 = E_+ \oplus \bar{E}_- \quad (4.11)$$

define complex, generalized, big-isotropic structures with the g -orthogonal bundles

$$E'_1 = E_1 \oplus S_c, \quad E'_2 = E_2 \oplus S_c \quad (S = S_+ \oplus S_-, \quad S_c = S \otimes \mathbb{C}). \quad (4.12)$$

The generalized metric F-structure defined by (γ, F_{\pm}, ψ) is said to be an integrable or a CRFK-structure if the following Courant bracket closure conditions are satisfied

$$\begin{aligned} [E_+, E_+] &\subseteq E_+, [E_+, S_{+c}] \subseteq E_+ \oplus S_{+c}, \\ [E_-, E_-] &\subseteq E_-, [E_-, S_{-c}] \subseteq E_- \oplus S_{-c}. \end{aligned} \quad (4.13)$$

Equivalently, a CRFK-structure is characterized by the fact that the structures E_1, E_2 are integrable and $[S_+, S_-] \subseteq S$ [13]. If F_{\pm} are complex structures $S = 0$ and the CRFK-structure is a generalized Kähler structure [2].

From (4.11), (4.12) and (4.13), we see that $(P = E_+, P' = E_+ \oplus S_{+c})$ might define a polarization of both E_1 and E_2 . But, the algebraic conditions included in (4.3) may not hold. Using (4.9), it follows that

$$\psi(F_{\pm}X, Y) + \psi(X, F_{\pm}Y) = 0 \quad (4.14)$$

implies $\omega|_{P \times P'} = 0$. Furthermore, it is a technical matter to check that, if the tensors $\psi \pm \gamma$ are non degenerate (e.g., if $\psi = t\psi'$ with $t \in \mathbb{R}_{\geq 0}$ small), then $TM \cap E_{1,2} = TM \cap E'_{1,2} = 0$. Hence, if ψ is such that $\psi \pm \gamma$ are non degenerate and (4.14) holds the subbundles P, P' define polarizations of E_1, E_2 . For a classical Kähler structure E_1 is determined by the antiholomorphic tangent bundle and E_2 is defined by the Kähler form. Classically, the antiholomorphic tangent bundle is regarded as a polarization of the symplectic structure defined by the Kähler form.

5 Appendix: Truncated cohomology

Truncated cohomology may also be useful in other situations, particularly in foliation theory.

Definition 5.1. Let $F \subseteq L$ be a pair of Lie algebroids over a manifold M . An *s-truncated form of degree k* ($0 \leq s \leq \dim L$) is a multilinear bundle morphism

$$\lambda : \underbrace{L \times \cdots \times L}_{s\text{-times}} \times \underbrace{F \times \cdots \times F}_{(k-s)\text{-times}} \rightarrow (M \times \mathbb{R}) \quad (5.1)$$

that is skew-symmetric with respect to all the arguments in F (even if their number is larger than $k - s$) and with respect to all the arguments in $L \setminus F$ separately.

We will denote by $\Omega_s^k(L, F)$ the space of s -truncated forms of degree k ; if $k > s + \dim F$ then $\Omega_s^k(L, F) = 0$. The truncated forms may be seen as restrictions of forms $\tilde{\lambda} \in \Omega^k(L) = \Gamma \wedge^k L^*$. Indeed, choose a complementary subbundle Q of F in L ($L = Q \oplus F$) and consider the decomposition

$$\wedge^k L^* = \oplus_{p+q=k} (\wedge^p Q^*) \otimes (\wedge^q F^*) \quad (k = 1, \dots, \text{rank } L). \quad (5.2)$$

Then, $\forall \lambda \in \Omega_s^k(L, F)$ we get uniquely defined forms $\lambda_{(p,q)}$ of type (p, q) ($p+q = k$) with $p \leq s$ by evaluating λ on p arguments in Q and q arguments in F and if

we take

$$\tilde{\lambda} = \lambda_{(0,k)} + \dots + \lambda_{(s,k-s)} \quad (5.3)$$

λ is the restriction of $\tilde{\lambda}$. Hence, if we denote $\Omega^{(p,q)}(Q, F) = \Gamma(\wedge^p Q^* \otimes \wedge^q F^*)$, we may identify

$$\Omega_s^k(L, F) \approx \tilde{\Omega}_s^k(Q, F) = \oplus_{p+q=k, p \leq s} \Omega^{(p,q)}(Q, F). \quad (5.4)$$

We define the exterior differential d_L^s of an s -truncated form by the same formula as the one used for the usual exterior differential d_L with the last $\geq k - s + 1$ arguments in ΓF . If $\lambda \in \Omega_s^k(L, F)$ then $d_L^s \lambda \in \Omega_s^{k+1}(L, F)$ and $(d_L^s)^2 \lambda = 0$ is a consequence of the Jacobi identity for L .

Thus, $(\Omega_s^k(L, F), d_L^s)$ is a cochain complex and defines cohomology spaces $H_s^k(L, F)$ that we call *s-intermediate de Rham cohomology spaces*. For $s = 0$ the result is the de Rham cohomology of F and for $s = \dim L$ the result is the de Rham cohomology of L . Restriction to arguments as required yields homomorphisms

$$h_{s>u} : H_s^k(L, F) \rightarrow H_u^k(L, F). \quad (5.5)$$

For the interpretation of the intermediate cohomology via (5.4) notice the following decomposition of the exterior differential d_L :

$$d_L = (d'_L)_{(1,0)} + (d''_L)_{(0,1)} + (\partial_L)_{(2,-1)}, \quad (5.6)$$

where the indices denote the (Q, F) -type of the operators and the following relations hold

$$\begin{aligned} (d''_L)^2 &= 0, d'_L d''_L + d'_L d''_L = 0, (\partial_L)^2 = 0, \\ d'_L \partial_L + \partial_L d'_L &= 0, (d'_L)^2 + d''_L \partial_L + \partial_L d''_L = 0 \end{aligned} \quad (5.7)$$

(see [7] for the case where $L = TM$ and F is a foliation on M).

Furthermore, put $\tilde{d}_L^s = \text{pr}_{\tilde{\Omega}_s^{k+1}(Q, F)} \circ d_L$, which means that for $\tilde{\lambda}$ defined by (5.3) one has

$$\tilde{d}_L^s \tilde{\lambda} = d_L \tilde{\lambda} - d'_L \lambda_{(s,k-s)} - \partial_L \lambda_{(s,k-s)} - \partial_L \lambda_{(s-1,k-s+1)}. \quad (5.8)$$

If we use (5.8) in order to compute $(\tilde{d}_L^s)^2$, (5.7) shows that everything cancels. Accordingly, $(\tilde{\Omega}_s^k(Q, F), \tilde{d}_L^s)$ is a cochain complex, which is isomorphic with the *s-intermediate de Rham complex* of the pair (L, F) and produces isomorphic cohomology spaces $H_s^k(L, F)$.

As an example, we prove a result for the intermediate cohomology of a foliation $F \subseteq L = TM$; in this case we shall omit the index L in the notation of the various differentials involved.

Proposition 5.1. *Let $\lambda \in \Omega_s^k(TM, F)$ be defined by (5.3) and assume that $d^s \lambda = 0$. Then, locally, if $k > s$ one has $\lambda = d^s \mu$ and if $k = s$ one has $\lambda = d\tilde{\mu} + \tilde{\nu}$ where $\tilde{\nu} \in \Gamma \wedge^s Q^*$ and $d'' \tilde{\nu} = 0$.*

Proof. In the proposition tilde has the significance given by formula (5.3). However, for simplicity we omit the sign tilde hereafter. We have a decomposition

$$d^s \lambda = \sum_{j=0}^s \xi_{(j,k+1-j)} \quad (5.9)$$

where

$$\xi_{(j,k+1-j)} = d'' \lambda_{(j,k-j)} + d' \lambda_{(j-1,k+1-j)} + \partial \lambda_{(j-2,k+2-j)} \quad (5.10)$$

and $d^s \lambda = 0$ is equivalent to $\xi_{(j,k+1-j)} = 0$ for $j = 0, \dots, s$.

If $k > s$ we can prove by induction that there are local forms μ such that

$$\lambda_{(j,k-j)} = d'' \mu_{(j,k-1-j)} + d' \mu_{(j-1,k-j)} + \partial \mu_{(j-2,k+1-j)}, \quad (j = 0, \dots, s). \quad (5.11)$$

For $j = 0$, (5.11) holds since

$$\xi_{(0,k+1)} = d'' \lambda_{(0,k)} = 0 \Rightarrow \lambda_{(0,k)} = d'' \mu_{(0,k-1)},$$

for a local form μ , in view of the Poincaré lemma for the operator d'' (e.g., [7]).

Then, if we assume that (5.11) holds for lower values of j , the annulation of the form given by (5.10) together with the properties (5.7) give

$$d''(\lambda_{(j,k-j)} + d' \mu_{(j-1,k-j)} + \partial \mu_{(j-2,k+1-j)}) = 0$$

and the d'' -Poincaré lemma produces the local form $\mu_{(j,k-1-j)}$ such that (5.11) holds for the index j .

Formulas (5.9), (5.10) applied to the form $\mu = \sum_{j=0}^s \mu_{(j,k-1-j)}$ yield the required conclusion for $k > s$.

If $k = s$, formula (5.11) similarly holds up to $j = s - 1$, which implies

$$\lambda = \lambda_{(s,0)} + d\mu - d' \mu_{(s-1,0)} - \partial \mu_{(s-2,1)}$$

for a local form $\mu = \sum_{h=1}^{s-1} \mu_{(h,s-h-1)}$. On the other hand, if we write down (5.10) for $j = s = k$ while inserting the values of $\lambda_{(s-1,1)}, \lambda_{(s-2,2)}$ given by (5.11) and use (5.7), we get

$$d''(\lambda_{(s,0)} - d' \mu_{(s-1,0)} - \partial \mu_{(s-2,1)}) = 0.$$

Therefore, if we denote

$$\nu = \lambda_{(s,0)} - d' \mu_{(s-1,0)} - \partial \mu_{(s-2,1)} \in \Gamma \wedge^s Q^*,$$

we exactly have the required conclusion. \square

Let $\Phi_s(F)$ be the sheaf of germs of forms of the type $d\mu + \nu$, $\mu \in \Omega^{s-1}(M)$, $\nu \in \wedge^s Q^*$, $d'' \nu = 0$ (i.e., ν is a basic s -form). Then, we get

Corollary 5.1. *The cohomology of M with values in $\Phi_s(F)$ is given by the formula*

$$H^u(M, \Phi_s(F)) = H_s^{s+u}(TM, F).$$

Proof. Proposition 5.1 shows that the sequence of sheaves

$$0 \rightarrow \Phi^s(F) \xrightarrow{\subseteq} \underline{\Omega}^s(M) \xrightarrow{d^s} \underline{\Omega}_s^{s+1}(TM, F) \xrightarrow{d^s} \dots$$

is a fine resolution of the sheaf $\Phi_s(F)$. \square

Proposition 5.1 and Corollary 5.1 hold for the more general case of pairs of Lie algebroids $F \subseteq L$ that satisfy the relative Poincaré lemma defined below.

Definition 5.2. 1) A Lie algebroid L *satisfies the Poincaré lemma* if, $\forall k \geq 0$, $\forall \lambda \in \Omega^{k+1}(L)$ with $d_L \lambda = 0$, there exist local forms $\mu \in \Omega^k(L)$ such that $\lambda = d_L \mu$. 2) A pair of Lie algebroids $F \subseteq L$ *satisfies the relative Poincaré lemma* if $\forall p \geq 0, q \geq 0$, $\forall \lambda \in \Omega^{(p,q+1)}(Q, F)$ with $d_L'' \lambda = 0$, there exist local forms $\mu \in \Omega^{(p,q)}(L)$ such that $\lambda = d_L'' \mu$.

If $F = L$ the relative Poincaré lemma is the Poincaré lemma for L . The relative Poincaré lemma is independent of the choice of Q since a change of Q only adds terms with more than $k - s$ arguments in F , which vanish. The operator d'' of a foliation satisfies the relative Poincaré lemma but, unfortunately, we know of no other interesting examples. (In the case of a complex manifold the sheaf Φ_s coincides with the sheaf of holomorphic s -forms and Proposition 5.1 gives nothing new.)

References

- [1] T. J. Courant, Dirac Manifolds, Transactions Amer. math. Soc., 319 (1990), 631-661.
- [2] M. Gualtieri, Generalized complex geometry, Ph.D. thesis, Univ. Oxford, 2003; arXiv:math.DG/0401221.
- [3] V. Itskov, M. Karasev and Yu. M. Vorobjev, Infinitesimal Poisson cohomology, A.M.S. Transl., 187 (2) (1998), 327-360.
- [4] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. I, Interscience Publ., New York, 1963.
- [5] B. Kostant, Quantization and unitary representations, Lect. Notes in Math. 170, Springer-Verlag, New York, 1970, 237-253.
- [6] J. M. Souriau, Structure des systèmes dynamiques, Dunod, Paris, 1969.
- [7] I. Vaisman, Cohomology and Differential Forms, M. Dekker, Inc., New York 1973.
- [8] I. Vaisman, Basic ideas of geometric quantization, Rend. Mat. Univ. Politecn. Torino, 37 (1979), 31-41.
- [9] I. Vaisman, Lectures on the geometry of Poisson manifolds, Progress in Math., vol. 118 Birkhäuser Verlag, Boston, 1994.

- [10] I. Vaisman, Dirac Structures and Generalized Complex Structures on $TM \times \mathbb{R}^h$, *Advances in Geom.* 7 (2007), 453-474.
- [11] I. Vaisman, Isotropic Subbundles of $TM \oplus T^*M$, *Int. J. of Geom. Methods in Modern Phys.*, 4 (3) (2007), 487-516.
- [12] I. Vaisman, Weak-Hamiltonian dynamical systems, *J. of Math. Phys.*, 48, 082903 (2007).
- [13] I. Vaisman, Generalized CRF-structures, *Geometriae Dedicata*, <http://dx.doi.org/10.1007/s10711-008-9239-z>; arXiv:0705.3934.
- [14] A. Wade, Conformal Dirac structures, *Lett. Math. Phys.* 53 (2000), 331-348.
- [15] A. Weinstein and M. Zambon, Variations on Prequantization, *Travaux mathématiques, Université de Luxembourg, Fascicule XVI (4th Conference on Poisson Geometry)*, pp. 187-219 (2005).

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